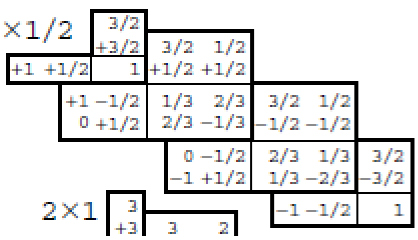
Clebsch-Gordan Coefficient Example
Phys 402

The eigenfunctions of J^2 can be expressed as linear combinations of states with different values of m_ℓ and m_s using the world-famous Clebsch-Gordan coefficients ($C^{\ell\ \ s\ \ j}_{m_\ell\ m_s\ m_j}$) as:

$$\left| j\ m_j \right\rangle = \sum_{m_\ell + m_s = m_j} C^{\ell\ \ s\ \ j}_{m_\ell\ m_s\ m_j} \left| \ell\ m_\ell \right\rangle \left| s\ m_s \right\rangle \qquad [4.185]$$

Where the ket $\left|\ell\ m_\ell\right\rangle$ represents the spherical harmonics $Y_\ell^{m_\ell}$. The C-G coefficient values are given in Table 4.8 on page 188 of Griffiths. Remember that all of the coefficients should appear under a square root, with the minus sign (if any) out front.

Now for an example of how to construct states that are simultaneous eigenfunctions of L^2, S^2, J^2 and J_z. Take the case again of hydrogen with $\ell = 1$ and spin $s = 1/2$. How do we find the state with $j = 3/2$ and $m_j = -1/2$ in terms of the $Y_\ell^{m_\ell}$ and spinors? Look at the $1\times 1/2$ CG Table on page 188. We are led to this table because we are combining an angular momentum vector with $\ell = 1$ and spin vector with $s = 1/2$.



Now look under the column labeled " $\begin{matrix}3/2\\-1/2\end{matrix}$ ". It says:

$$\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle = \sum_{m_\ell + m_s = -1/2} C^{1\ \ 1/2\ \ 3/2}_{m_\ell\ \ m_s\ \ -1/2} \left|1\ m_\ell\right\rangle \left|\frac{1}{2}\ m_s\right\rangle$$

$$\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1\ 0\right\rangle\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|1\ -1\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$$

This can be written in a more familiar way in terms of spherical harmonics and spinors as:

$$\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}Y_1^0\chi_- + \sqrt{\frac{1}{3}}Y_1^{-1}\chi_+$$

One can move back and forth between the coupled and un-coupled representations using the Clebsch-Gordan table on page 188. Here is the schematic layout for the CG table for combining two spins (called $\vec{S}_1, \vec{S}_2$) to form a total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ (S^2 has eigenvalue $s(s+1)\hbar^2$):

General Schematic of the C-G Table

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$S_1 \times S_2$$

$$\begin{array}{|c|} \hline s \\ m_s \end{array}$$

Coupled Representation

$$\begin{array}{|cc} \hline m_{s_1} & m_{s_2} \\ \hline \end{array} \qquad CG\#$$

Un-Coupled Representation